Problem 3

Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a+b) = f(a)f(b)$ for all real numbers a and b. Show that $f'(x) = f(x)$ for all x and deduce that $f(x) = e^x$.

Solution

By definition, the first derivative is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Now make use of the property $f(a+b) = f(a)f(b)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h}$$
$$f'(x) = f(x) \lim_{h \to 0} \frac{f(h) - 1}{h}$$

Since $f(0) = 1$, this is an indeterminate form (0/0), and L'Hôpital's rule can be applied. Differentiate the top and bottom with respect to h.

$$f'(x) = f(x) \lim_{h \to 0} \frac{f'(h) - 0}{1}$$
$$f'(x) = f(x)f'(0)$$

Since $f'(0) = 1$, we have the desired result.

$$f'(x) = f(x)$$

Now separate variables.

$$\frac{df}{dx} = f$$
$$\frac{df}{f} = dx$$
$$\ln|f| = x + C$$
$$|f| = e^{x+C}$$
$$f(x) = Ae^x$$

$f(0) = 1$ implies that $f(0) = A = 1$. Therefore,

$$f(x) = e^x.$$